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The Shift-Invariant Subspaces in $L_1(\mathbf{R})$[1]

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In this paper, we investigate finitely generated shift-invariant subspaces in $L_1(\mathbf{R})$. We first introduce the notion of the convolution of a vector sequence and a matrix sequence. Then by the theory of dual space of the normed linear space we obtain the complete characterizations of finitely generated shift-invariant spaces in $L_1(\mathbf{R})$, based on the existence of generator with linearly independent shifts in finitely generated shift-invariant subspaces on the real line. © 2002 Elsevier Science (USA)

Key Words: shift-invariant spaces; linearly independent shifts; generator; annihilator.

1. INTRODUCTION

The shift-invariant spaces generated by a finite number of compactly supported functions play important role in wavelet theory and other analysis area. A linear space of functions S on $\mathbf{R}$ is said to be shift-invariant if $f(\cdot - k) \in S$ for any $k \in \mathbf{Z}$ whenever $f \in S$. Given a finite collection $\Phi = \{\phi_1, \phi_2, \ldots, \phi_m\}$ of compactly supported functions, $S_0(\Phi)$ denotes the linear span of $\{\phi_i(\cdot - j) : i = 1, 2, \ldots, m; j \in \mathbf{Z}\}$. It is obvious that $S_0(\Phi)$ is the smallest shift-invariant space containing Φ. If Φ is a subset of $L_p(\mathbf{R})$, we use $S_p(\Phi)$ to denote the closure of $S_0(\Phi)$ in $L_p(\mathbf{R})$. Here, as usual, $L_p(\mathbf{R})$ is the normed linear space of all measurable functions $f(x)$ on $\mathbf{R}$ such that $|f(x)|^p$ is integrable and the norm is defined by

$$\|f\|_p := \left(\int_{-\infty}^{\infty} |f(x)|^p \, \mathrm{d}x \right)^{1/p}.$$

Let $l(\mathbf{Z})$ be the linear space of all sequences on $\mathbf{Z}$, and $l_0(\mathbf{Z})$ the linear subspace consisting of all finitely supported sequences on $\mathbf{Z}$. Given a sequence $c \in l(\mathbf{Z})$, we define $\|c\|_p := (\sum_\alpha |c(\alpha)|^p)^{1/p}$, $1 \leqslant p < \infty$, $\|c\|_\infty :=$

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$\sup_\alpha\{|c(\alpha)|\}$. For $1 \leqslant p \leqslant \infty$, $l_p(\mathbf{Z})$ denotes the normed linear space of all sequences c on $\mathbf{Z}$ such that $||c||_p < \infty$. For $b, c \in l(\mathbf{Z})$ the convolution of b and c is defined by

$$b * c(\alpha) = \sum_\beta b(\alpha - \beta)c(\beta), \qquad \alpha \in \mathbf{Z},$$

whenever the above series is absolutely convergent. For a given $c \in l(\mathbf{Z})$ the symbol of c is the formal series $\sum_\alpha c(\alpha)z^{-\alpha}$ denoted by $\tilde{c}(z)$. We shall use the semi-convolution to study the structure of $S_p(\Phi)$. For a given function ϕ and a sequence $c \in l(\mathbf{Z})$ the semi-discrete convolution $\phi *' c$ is defined as the sum

$$\sum_\alpha \phi(\cdot - \alpha)c(\alpha)$$

which is well defined under various assumptions on the function ϕ and the sequence c. It is obvious that this sum makes sense if ϕ has compact support. $S(\Phi)$ denotes the linear space of all functions $\sum_{i=1}^m \phi_i *' c_i$ where $c = (c_1, \ldots, c_m) \in (l(\mathbf{Z}))^m$. The linear space $S(\Phi)$ is called finitely generated by Φ. For a subspace A of $(l(\mathbf{Z}))^m$ it corresponds a subspace $S_A(\Phi)$ of $S(\Phi)$ defined as

$$S_A(\Phi) = \left\{ f : f = \sum_{i=1}^m \phi_i *' a_i, a = (a_1, \ldots, a_m) \in A \right\}.$$

In [1, 2] de Boor, Devore and Ron studied finitely generated shift-invariant subspaces of $L_2(\mathbf{R}^s)$, and gave a characterization for such spaces in terms of the Fourier transforms of their generators. In [5] Jia studied shift-invariant spaces generated by a finite number of compactly supported functions in $L_p(\mathbf{R}^s)$ $(1 \leqslant p \leqslant \infty)$, and gave a characterization of such spaces in terms of the semi-convolutions of their generators with sequences on $\mathbf{Z}^s$. When $s = 1$, and $1 < p < \infty$, without the stability assumption the following theorem was obtained by Jia [4].

THEOREM. *For* $1 < p < \infty$

$$S_p(\Phi) = S(\Phi) \cap L_p(\mathbf{R}). \tag{1}$$

Consequently for $1 < p < \infty$ *a function* $f \in L_p(\mathbf{R})$ *lies in* $S_p(\Phi)$ *if and only if there are sequences* a_i $(i = 1, \ldots, m)$ *such that*

$$f = \sum_{i=1}^m \sum_\alpha \phi_i(\cdot - \alpha)a_i(\alpha).$$

However, when $p = 1$ the situation is completely different. Generally, the conclusion of the above theorem is not valid in the space $L_1(\mathbf{R})$. Here is one example.

EXAMPLE. Let χ be the characteristic function of [0,1), and $\psi = \chi - \chi(\cdot - 1)$. It is obvious that $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 0$ for any function f in $S_0(\psi)$. We also have $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 0$ for any $f \in S_1(\psi)$ since $S_1(\psi)$ is the closure of $S_0(\psi)$ in $L_1(\mathbf{R})$. However $\int_{-\infty}^{\infty} \chi(x)\,\mathrm{d}x = 1$. This means that $\chi \notin S_1(\psi)$. It is easy to verify that $\chi = \sum_{j=0}^{\infty} \psi(\cdot - j) \in S(\psi)$. Therefore, $S_1(\psi) \neq S(\psi) \cap L_1(\mathbf{R})$.

The main purpose of this paper is to give the characterizations of finitely generated shift-invariant spaces in $L_1(\mathbf{R})$. After introducing the convolution of a vector sequence and a matrix sequence, we study the characterization of $S_1(\Phi)$, and we obtain the sufficient and necessary conditions which guarantee $S_1(\Phi) = S(\Phi) \cap L_1(\mathbf{R})$ by the theory of the dual space of the normed linear space $L_1(\mathbf{R})$.

It is worth noting that when $s > 1$ it was proved [5] that (1) is true under the extra assumption that Φ consists of a finite number of compactly supported functions in $L_p(\mathbf{R}^s)$ whose shifts are stable.

2. CHARACTERIZATIONS OF $S_1(\Phi)$

To treat shift-invariant spaces the generators with linearly independent shifts play important role. Let $\{f_1, \ldots, f_k\}$ be a finite collection of compactly supported functions on $\mathbf{R}$ and $a_j = \{a_j(\alpha)\} \in l(\mathbf{Z})$ $(j = 1, \ldots, k)$, the shifts of $\{f_1, \ldots, f_k\}$ are said to be linearly independent if

$$\sum_{j=1}^{k} \sum_{\alpha} a_j(\alpha) f_j(x - \alpha) = 0 \Rightarrow a_j = 0, \qquad j = 1, \ldots, k.$$

If, in addition, $\{f_1, \ldots, f_k\}$ is a finite collection of compactly supported functions in $L_p(\mathbf{R})$, we say that the shifts of $\{f_1, \ldots, f_k\}$ are L_p-stable if there exist two positive constants C_1 and C_2 such that for any sequences $a_1, \ldots, a_k \in l_p(\mathbf{Z})$,

$$C_1 \sum_{j=1}^{k} ||a_j||_p \leqslant \left|\left| \sum_{j=1}^{k} f_j *' a_j \right|\right|_p \leqslant C_2 \sum_{j=1}^{k} ||a_j||_p.$$

It was proved by Jia and Micchelli [6] that if shifts of $\{f_1, \ldots, f_k\}$ are linearly independent, they are L_p-stable.

For the existence of the generator with linearly independent shifts in $S(\Phi)$ we appeal for the following result which was proved in [4]:

THEOREM(J). *Let Φ be a finite collection of nontrivial distributions on $\mathbf{R}$ with compact support. Then there exists another finite collection Ψ of compactly supported distributions on $\mathbf{R}$ with following properties*:

(1) *The shifts of the elements in Ψ are linearly independent*;

(2) $\#\Psi \leqslant \#\Phi$;

(3) $\Phi \subset S_0(\Psi)$;

(4) $S(\Psi) = S(\Phi)$.

If, in addition, $\Phi \subset L_p(\mathbf{R})$ for some p $(1 \leqslant p \leqslant \infty)$, then Ψ can be chosen to be a subset of $L_p(\mathbf{R})$.

To describe the characterizations of $S_1(\Phi)$ we introduce the notion of the convolution of a vector sequence and a matrix sequence. Given sequences $c_{ij} = \{c_{ij}(\alpha)\} \in l_0(\mathbf{Z})$ $(i = 1, \dots, l; j = 1, \dots, k)$, set $C = (c_{ij})_{1 \leqslant i \leqslant l; 1 \leqslant j \leqslant k} \in (l_0(\mathbf{Z}))^{l \times k}$. For $f = (f_1, \dots, f_l) \in (l_0(\mathbf{Z}))^l$ define $h = (h_1, \dots, h_k) \in (l_0(\mathbf{Z}))^k$ by

$$h_j(\alpha) = \sum_{i=1}^{l} \sum_{\beta} f_i(\beta) c_{ij}(\alpha - \beta) \quad (j = 1, \dots, k), \quad \alpha \in \mathbf{Z}.$$

We call h as the convolution of the vector sequence f and the matrix sequence C, and denoted by $h = f * C$. Let

$$M_C = \{h = (h_1, \dots, h_k) : h = f * C, f \in (l_0(\mathbf{Z}))^l\},$$

and $\bar{M}_C$ denote the closure of M_C in normed linear space $(l_1(\mathbf{Z}))^k$.

Let $\Phi = \{\phi_1, \phi_2, \dots, \phi_m\}$ be a finite collection of compactly supported integrable functions. As a consequence of Theorem(J) in $S(\Phi)$ we can find a generator with L_1-stable shifts the elements of which have compact supports.

THEOREM 1. *Assume that $\Theta = \{\theta_1, \dots, \theta_k\}$ the elements of which have compact supports is the generator with L_1-stable shifts in $S(\Phi)$, and*

$$\phi_i = \sum_{j=1}^{k} \sum_{\alpha} r_{ij}(\alpha) \theta_j(\cdot - \alpha) \quad (i = 1, \dots, m).$$

Let $R = (r_{ij})_{1 \leqslant i \leqslant m; 1 \leqslant j \leqslant k} \in (l_0(\mathbf{Z}))^{m \times k}$. Then $S_1(\Phi) = S_{\bar{M}_R}(\Theta)$; $S_1(\Phi) = S(\Phi) \cap L_1(\mathbf{R})$ if and only if $\bar{M}_R = (l_1(\mathbf{Z}))^k$.

Proof. If $f \in S_1(\Phi)$, there exists some sequence $\{F_l(x)\}$ of $S_0(\Phi)$ such that

$$\lim_{l \to \infty} ||F_l - f||_1 = 0.$$

Assume

$$F_l(x) = \sum_{i=1}^{m} \sum_{\alpha} f_i^{(l)}(\alpha)\phi_i(x-\alpha),$$

where $\{f_i^{(l)}(\beta)\} \in l_0(\mathbf{Z})$ $(i = 1, \ldots, m; l = 1, 2, \ldots)$. We have

$$\begin{aligned} F_l(x) &= \sum_{i=1}^{m} \sum_{\alpha} f_i^{(l)}(\alpha)\phi_i(x-\alpha) \\ &= \sum_{i=1}^{m} \sum_{\alpha} f_i^{(l)}(\alpha)\left(\sum_{j=1}^{k} \sum_{\beta} r_{ij}(\beta)\theta_j(x-\alpha-\beta)\right) \\ &= \sum_{j=1}^{k} \sum_{\gamma} \sum_{i=1}^{m} \sum_{\alpha} f_i^{(l)}(\alpha) r_{ij}(\gamma-\alpha)\theta_j(x-\gamma) \\ &= \sum_{j=1}^{k} \sum_{\gamma} d_j^{(l)}(\gamma)\theta_j(x-\gamma), \end{aligned}$$

where $d_j^{(l)}(\gamma) = \sum_{i=1}^{m} \sum_{\alpha} f_i^{(l)}(\alpha) r_{ij}(\gamma-\alpha)$. Obviously, $d^{(l)} = (d_1^{(l)}, \ldots, d_k^{(l)}) \in M_R$. Since the L_1-stability of $\{\theta_1, \ldots, \theta_k\}$ it follows that the convergence of $\{F_l(x)\}$ in L_1 is equivalent to the convergence of the sequences $d_j^{(l)}$ $(j = 1, \ldots, k)$ in $l_1(\mathbf{Z})$. Consequently, there exist the sequences $d_1, \ldots, d_k$ in $l_1(\mathbf{Z})$ such that

$$\lim_{l\to\infty} ||d_j^{(l)} - d_j||_1 = 0, \qquad j = 1, \ldots, k$$

and

$$\lim_{l\to\infty} \left|\left| F_l(\cdot) - \sum_{j=1}^{k} \sum_{\gamma} d_j(\gamma)\theta_j(\cdot-\gamma) \right|\right|_1 = 0.$$

Hence

$$f(x) = \sum_{j=1}^{k} \sum_{\gamma} d_j(\gamma)\theta_j(x-\gamma).$$

Since $d = (d_1, \ldots, d_k) \in \bar{M}_R$, we have $f(x) \in S_{\bar{M}_R}(\Theta)$. Conversely, assume $f(x) \in S_{\bar{M}_R}(\Theta)$, i.e.,

$$f(x) = \sum_{j=1}^{k} \sum_{\gamma} e_j(\gamma)\theta_j(x-\gamma),$$

where $e=(e_1,\dots,e_k)\in\bar{M}_R$. By the definition of $\bar{M}_R$, we can find sequences $g^{(l)}=(g_1^{(l)},\dots,g_k^{(l)})$ and $f^{(l)}=(f_1^{(l)},\dots,f_m^{(l)})\in(l_0(\mathbf{Z}))^m$ such that

$$g^{(l)}=f^{(l)}*R,$$

$$\lim_{l\to\infty}||g_j^{(l)}-e_j||_1=0,\quad (j=1,\dots,k).$$

Define the function sequence $G_l(x)(l=1,2,\dots)$ as

$$G_l(x)=\sum_{j=1}^{k}\sum_{\alpha}g_j^{(l)}(\alpha)\theta_j(x-\alpha).$$

Since $\Theta=\{\theta_1,\dots,\theta_k\}$ has L_1 stable shifts, from

$$G_l(x)-f(x)=\sum_{j=1}^{k}\sum_{\alpha}(g_j^{(l)}(\alpha)-e_j(\alpha))\theta_j(x-\alpha),$$

we have $\lim_{l\to\infty}||G_l(x)-f(x)||_1=0$. However,

$$\begin{aligned}G_l(x)&=\sum_{j=1}^{k}\sum_{\alpha}g_j^{(l)}(\alpha)\theta_j(x-\alpha)\\&=\sum_{j=1}^{k}\sum_{\alpha}\sum_{i=1}^{m}\sum_{\beta}f_i^{(l)}(\beta)r_{ij}(\alpha-\beta)\theta_j(x-\alpha)\\&=\sum_{i=1}^{m}\sum_{\beta}f_i^{(l)}(\beta)\sum_{j=1}^{k}\sum_{\alpha}r_{ij}(\alpha-\beta)\theta_j(x-\alpha)\\&=\sum_{i=1}^{m}\sum_{\beta}f_i^{(l)}(\beta)\sum_{j=1}^{k}\sum_{\gamma}r_{ij}(\gamma)\theta_j(x-\beta-\gamma)\\&=\sum_{i=1}^{m}\sum_{\beta}f_i^{(l)}(\beta)\phi_i(x-\beta).\end{aligned}$$

This verifies $G_l(x)\in S_0(\Phi)$. We obtain $f(x)\in S_1(\Phi)$ and have proved $S_1(\Phi)=S_{\bar{M}_R}(\Theta)$.

Since the generator $\Theta=\{\theta_1,\dots,\theta_k\}$ has L_1-stable shifts, it is well known (see for example [5]) that $S(\Theta)\cap L_1(\mathbf{R})=S_{(l_1(\mathbf{Z}))^k}(\Theta)$. Noting $S(\Theta)=S(\Phi)$ we obtain $S_1(\Phi)=S(\Phi)\cap L_1(\mathbf{R})$ if and only if $\bar{M}_R=(l_1(\mathbf{Z}))^k$. ∎

It is well known that the set of all continuous linear functionals on the normed linear space $(l_1(\mathbf{Z}))^k$ is $(l_\infty(\mathbf{Z}))^k$ called as the dual space of $(l_1(\mathbf{Z}))^k$. For any $c=(c_1,\dots,c_k)\in(l_\infty(\mathbf{Z}))^k$ and $d=(d_1,\dots,d_k)\in(l_1(\mathbf{Z}))^k$ we denote by $\langle c, d\rangle$ the value of the functional c at d. We have

$$\langle c, d\rangle = \sum_{i=1}^{k} \langle c_i, d_i\rangle = \sum_{i=1}^{k} \sum_{\alpha} c_i(\alpha) d_i(\alpha).$$

Using the theory of the dual space, we shall prove the following theorem which describes the conditions guaranteeing $\bar{M}_R = (l_1(\mathbf{Z}))^k$. Let T denote the unit circle $\{z : |z| = 1\}$ on the complex plane.

THEOREM 2. *Assume that* $R = (r_{ij})_{1 \leqslant i \leqslant m; 1 \leqslant j \leqslant k} \in (l_0(\mathbf{Z}))^{m \times k}$, $\tilde{r}_{ij}(z) = \sum_{\alpha} r_{ij}(\alpha) z^{-\alpha}$ *and the matrix* $\tilde{R}(z) = (\tilde{r}_{ij}(z))_{1 \leqslant i \leqslant m; 1 \leqslant j \leqslant k}$. *Then* $\bar{M}_R = (l_1(\mathbf{Z}))^k$ *if and only if* $\tilde{R}(z)$ *has rank* k *for any* $z \in T$.

Proof. Assume that there exists $\lambda \in T$ such that the rank of $\tilde{R}(\lambda) < k$. There exists $(t_1, \ldots, t_k) \neq 0$ such that

$$t_1 \tilde{r}_{i1}(\lambda) + \cdots + t_k \tilde{r}_{ik}(\lambda) = 0, \qquad i = 1, \ldots, m.$$

Define $s = (s_1, \ldots, s_k) \in (l(\mathbf{Z}))^k$ as follows:

$$s_1(\alpha) = t_1 \lambda^{-\alpha}, \quad s_2(\alpha) = t_2 \lambda^{-\alpha}, \quad \ldots, s_k(\alpha) = t_k \lambda^{-\alpha}, \quad \alpha \in \mathbf{Z}.$$

It is obvious $s \in (l_\infty(\mathbf{Z}))^k$, and s is nontrivial. Assume $g = (g_1, \ldots, g_k) \in M_R$. There is $f = (f_1, \ldots, f_m) \in (l_0(\mathbf{Z}))^m$ such that

$$g_j(\beta) = \sum_{i=1}^{m} \sum_{\alpha} f_i(\alpha) r_{ij}(\beta - \alpha), \qquad j = 1 \ldots, k, \quad \beta \in \mathbf{Z}.$$

We have

$$\begin{aligned}
\langle s, g\rangle &= \sum_{j=1}^{k} \langle s_j, g_j\rangle = \sum_{j=1}^{k} \sum_{\beta} g_j(\beta) s_j(\beta) = \sum_{j=1}^{k} \sum_{\beta} g_j(\beta) t_j \lambda^{-\beta} \\
&= \sum_{j=1}^{k} \sum_{\beta} \sum_{i=1}^{m} \sum_{\alpha} f_i(\alpha) t_j \lambda^{-\beta} r_{ij}(\beta - \alpha) \\
&= \sum_{j=1}^{k} \sum_{i=1}^{m} \sum_{\alpha} f_i(\alpha) \sum_{\gamma} t_j \lambda^{(-\gamma-\alpha)} r_{ij}(\gamma) \\
&= \sum_{i=1}^{m} \sum_{\alpha} (f_i(\alpha) \lambda^{-\alpha}) \sum_{j=1}^{k} \sum_{\gamma} t_j \lambda^{-\gamma} r_{ij}(\gamma) = 0.
\end{aligned}$$

Hence for any $d=(d_1,\dots,d_k)\in\bar{M}_R$ by the continuity of the functional s we have

$$\langle s,d\rangle=\sum_{j=1}^{k}\langle s_j,d_j\rangle=0.$$

By the completeness of the normed linear space $(l_1(\mathbf{Z}))^k$, we obtain $\bar{M}_R\neq(l_1(\mathbf{Z}))^k$.

On the other hand if $\tilde{R}(z)$ has rank k for any $z\in T$, we shall prove $\bar{M}_R=(l_1(\mathbf{Z}))^k$. We set $u_{lj}(\alpha)=\bar{r}_{jl}(-\alpha)$ $\alpha\in\mathbf{Z}$, $1\leqslant j\leqslant m$; $1\leqslant l\leqslant k$; $\tilde{u}_{lj}(z)=\sum_\alpha u_{lj}(\alpha)z^{-\alpha}$; $U(z)=(\tilde{u}_{lj}(z))_{1\leqslant l\leqslant k;1\leqslant j\leqslant m}$. Now the $k\times k$ matrix $M(z)=U(z)\tilde{R}(z)$ has rank k for every $z\in T$. Let $M_{lj}(z)$ denote the cofactor of the entry $m_{lj}(z)$ of the matrix $M(z)=(m_{lj}(z))_{1\leqslant l\leqslant k;1\leqslant j\leqslant k}$. Since $\det M(z)\neq 0$ for any $z\in T$ we may set

$$\tilde{n}_{lj}(z)=\frac{M_{jl}(z)}{\det M(z)},\quad l=1,\dots,k;\quad j=1,\dots,k$$

for $z\in T$. The matrix $N(z)=(\tilde{n}_{lj}(z))_{1\leqslant l\leqslant k;1\leqslant j\leqslant k}$ satisfies $N(z)M(z)=I$ for every $z\in T$, where I is $k\times k$ identity matrix. It is known that there are expansions $\tilde{n}_{lj}(z)=\sum_\alpha n_{lj}(\alpha)z^{-\alpha}$ where the sequences $n_{lj}=\{n_{lj}(\alpha)\}_\alpha$ are all exponentially decay. We need to prove $(l_1(\mathbf{Z}))^k\subseteq\bar{M}_R$. To this end for any $d=(d_1,\dots,d_k)\in(l_1(\mathbf{Z}))^k$ set

$$g_j=\sum_{l=1}^{k}d_l*n_{lj}\qquad(j=1,\dots,k),$$

$$f_l=\sum_{j=1}^{k}g_j*u_{jl}\qquad(l=1,\dots,m).$$

It is easy to verify that $d_j=\sum_{l=1}^m f_l*r_{lj},\ (j=1,\dots,k)$. Let

$$f_l^{(s)}(\beta)=\begin{cases}f_l(\beta), & |\beta|\leqslant s,\\ 0, & |\beta|>s\end{cases}$$

and $d_j^{(s)}=\sum_{l=1}^m f_l^{(s)}*r_{lj}$. We have $\lim_{s\to\infty}||f_l^{(s)}-f_l||_1=0$. Consequently,

$$\begin{aligned}||d_j^{(s)}-d_j||_1&=\left\|\sum_{j=1}^{m}f_l*r_{lj}-\sum_{j=1}^{m}f_l^{(s)}*r_{lj}\right\|_1\\&\leqslant\sum_{l=1}^{m}||f_l-f_l^{(s)}||_1\max_{l,j}||r_{lj}||_1.\end{aligned}$$

We obtain $\lim_{s\to\infty} ||d_j^{(s)} - d_j||_1 = 0$ $(j = 1, \ldots, k)$. Obviously $d^{(s)} = (d_1^{(s)}, \ldots, d_k^{(s)}) \in M_R$ and $d^{(s)}$ converges to d in the space $(l_1(\mathbf{Z}))^k$, this shows $d \in \bar{M}_R$. Hence $(l_1(\mathbf{Z}))^k \subseteq \bar{M}_R$. ∎

For understanding the structure of $S_1(\Phi)$ it is important to investigate the characterization of the subspace $\bar{M}_R$ of $(l_1(\mathbf{Z}))^k$ since $S_1(\Phi) = S_{\bar{M}_R}(\Theta)$ as described in Theorem 1. For this purpose, we introduce some notations in the dual space further. The theory of the dual space can also be used to investigate the stability of the shifts of distributions [9]. For a normed vector space H the set of all continuous linear functionals on H is denoted by H'. Let X be a subset of H, a functional $x' \in H'$ is called an annihilator of X if

$$x'(x) = 0, \qquad x \in X.$$

The set of all annihilators of X is denoted by X°. On the other hand for a subset Y of H' we call $x \in H$ an annihilator of Y if

$$x'(x) = 0, \qquad x' \in Y.$$

The set of such annihilators of Y is denoted by ${}^\circ Y$. The following result is easily derived [8].

LEMMA 1. *Let H be a normed vector space. If X is a subset of H, and M is the closed subspace spanned by X in H, then $M = {}^\circ(X^\circ)$.*

Therefore, we have $\bar{M}_R = {}^\circ(M_R^\circ)$. Furthermore, we desire to emphasize that M_R° is related to the solutions to a system of homogeneous linear difference equations. To this end for a given $\alpha \in \mathbf{Z}$, τ^α denotes the difference operator on $l(\mathbf{Z})$, which is defined by

$$\tau^\alpha b := b(\cdot + \alpha), \qquad b \in l(\mathbf{Z}).$$

A Laurent polynomial $P(z) = \sum_\alpha d(\alpha) z^\alpha$ induces a difference operator

$$P(\tau) = \sum_\alpha d(\alpha)\tau^\alpha.$$

From an element $f = (f_1, \ldots, f_m) \in (l_0(\mathbf{Z}))^m$ we obtain an element $h = (h_1, \ldots, h_k)$ of M_R, where $h = (h_1, \ldots, h_k) \in (l(\mathbf{Z}))^k$ and

$$h_i(\alpha) = \sum_{j=1}^m \sum_\beta f_j(\beta) r_{ji}(\alpha - \beta), \qquad \alpha \in \mathbf{Z},\ i = 1, \ldots, k.$$

Assume that $s = (s_1, \ldots, s_k) \in (l_\infty(\mathbf{Z}))^k$ is in M_R°. Then

$$\langle s, h\rangle = \sum_{i=1}^{k} \sum_{\alpha} s_i(\alpha) h_i(\alpha)$$

$$= \sum_{i=1}^{k} \sum_{\alpha} s_i(\alpha) \sum_{j=1}^{m} \sum_{\beta} f_j(\beta) r_{ji}(\alpha - \beta)$$

$$= \sum_{j=1}^{m} \sum_{\beta} f_j(\beta) \sum_{i=1}^{k} \sum_{\mu} r_{ji}(\mu) s_i(\mu + \beta) = 0. \tag{2}$$

Because (2) holds for every $f = (f_1, \ldots, f_m) \in (l_0(\mathbf{Z}))^m$, we obtain

$$\sum_{i=1}^{k} \sum_{\mu} r_{ji}(\mu) s_i(\mu + \beta) = 0, \qquad \beta \in \mathbf{Z}, \quad j = 1, \ldots, m.$$

This means that $s = (s_1, \ldots, s_k)$ must satisfy the system of difference equations:

$$\sum_{i=1}^{k} \tilde{r}_{ji}(\tau) s_i = 0, \qquad j = 1, \ldots, m. \tag{3}$$

As before $\tilde{R}(z) = (\tilde{r}_{ij}(z))_{1 \leqslant i \leqslant m; 1 \leqslant j \leqslant k}$, let

$$\Omega(\tilde{R}) = \{\omega \in \mathbf{C} \backslash \{0\} : \operatorname{rank}(\tilde{R}(\omega)) < k\}.$$

It can be proved (see [7]) that all solutions to the system of difference equations (3) form a finite-dimensional subspace of $(l(\mathbf{Z}))^k$, which is denoted by $\tau(\tilde{R})$, and every element $f = (f_1, \ldots, f_k) \in \tau(\tilde{R})$ has the form

$$f_j(\alpha) = \sum_{\omega \in \Omega(\tilde{R})} \omega^\alpha q_{j,\omega}(\alpha), \qquad \alpha \in \mathbf{Z}, \tag{4}$$

where $q_{j,\omega}$ are polynomials.

LEMMA 2. *The annihilator $M_R{}^\circ$ of M_R is finite dimensional.*

(1°) $M_R{}^\circ = \tau(\tilde{R}) \cap (l_\infty(\mathbf{Z}))^k$;

(2°) *if $\Omega(\tilde{R}) \cap T = \{t_1, \ldots, t_L\}$ is not empty, every element $s = (s_1, \ldots, s_k) \in M_R{}^\circ$ has the form*

$$s_j(\alpha) = \sum_{\nu=1}^{L} c_{j\nu} t_\nu^\alpha.$$

Proof. Condition (1°) is derived from (3) and the definition of M_R°. From (4) it is obvious that an element $s = (s_1, \dots, s_k)$ of $\tau(\tilde{R})$ lies in $(l_\infty(\mathbf{Z}))^k$ if and only if sum (4) takes $\omega \in \Omega(\tilde{R}) \cap T$ and constant polynomials as $q_{j,\omega}$. This gives (2°). ∎

Now suppose $\{h_1, \dots, h_N\}$ is a base of M_R° and $h_j = (h_{j1}, \dots, h_{jk}) \in (l_\infty(\mathbf{Z}))^k$ has the form

$$h_{jl}(\alpha) = \sum_{v=1}^{L} c_{jlv} t_v^\alpha, \tag{5}$$

where c_{jlv} are complex, $t_v \in \Omega(\tilde{R}) \cap T$.

THEOREM 3. *Assume $\Theta = \{\theta_1, \dots, \theta_k\}$ is the generator with L_1-stable shifts in $S(\Phi)$, $\{h_1, \dots, h_N\}$ is a base of M_R°, and h_j has the form* (5). *Then*

$$S_1(\Phi) = \left\{ f(x) = \sum_{l=1}^{k} \theta_l *' a_l : (a_1, \dots, a_k) \in (l_1(\mathbf{Z}))^k, \right.$$
$$\left. \sum_{l=1}^{k} \sum_{\alpha} \sum_{v=1}^{L} t_v^\alpha c_{jlv} a_l(\alpha) = 0, \quad j = 1, \dots, N \right\}. \tag{6}$$

Proof. From Theorem 1 we know

$$S_1(\Phi) = \left\{ f(x) = \sum_{l=1}^{k} \theta_l *' a_l : (a_1, \dots, a_k) \in \bar{M}_R \right\}.$$

Considering that $\{h_1, \dots, h_N\}$ is a base of M_R°, conclusion (6) is obtained from $\bar{M}_R = {}^\circ(M_R^\circ)$. ∎

EXAMPLE. Let us study the previous example on the basis of the results of this paper. Now $\Psi = \{\psi\}$. From $\psi(\cdot) = \chi(\cdot) - \chi(\cdot - 1)$ we know $\tilde{R}(z) = 1 - z^{-1}$. For $z_0 = 1$ we have $\tilde{R}(z_0) = 0$. Thus, we obtain $\bar{M}_R \neq l_1(\mathbf{Z})$ and $S_1(\Psi) \neq S(\Psi) \cap L_1(\mathbf{R})$ by Theorems 1 and 2. This coincides with our direct observation previously. It is obvious that in this case M_R° has a base $\{s\}$ where $s = \{s(\alpha)\}_{\alpha \in \mathbf{Z}}$ and $s(\alpha) = 1, \ \alpha \in \mathbf{Z}$. Hence by Theorem 3

$$S_1(\Psi) = \left\{ \sum_{\alpha} \chi(\cdot - \alpha) a(\alpha) : \sum_{\alpha} |a(\alpha)| < \infty \text{ and } \sum_{\alpha} a(\alpha) = 0 \right\}. \tag{7}$$

We can also verify (7) directly. In fact, if $f(x) \in S_1(\Psi)$, we know $f(x)$ can be written as $f(x) = \sum_\alpha a(\alpha)\chi(\cdot - \alpha)$ with $\sum |a(\alpha)| < \infty$ since $S_1(\Psi) \subseteq S(\Psi) \cap L_1(\mathbf{R})$ [5]. Moreover, it holds $\int f(x)\,\mathrm{d}x = 0$ as we have observed. This induces $\sum a(\alpha) = 0$. Conversely, if $\sum |a(\alpha)| < \infty$ and $\sum_\alpha a(\alpha) = 0$, it is obvious that the function $f(x) = \sum_\alpha \chi(\cdot - \alpha)a(\alpha)$ lies in $L_1(\mathbf{R})$. For positive integers M, N and an integer k define

$$A(k) = \sum_{\alpha=-\infty}^{k} a(\alpha),$$

$$f_{M,N} = \sum_{\alpha=-M}^{N} \psi(\cdot - \alpha)A(\alpha).$$

It is obvious that $f_{M,N}$ lies in $S_0(\Psi)$. We have

$$f(x) - f_{M,N}(x) = \sum_{\alpha=-\infty}^{-M} \chi(x-\alpha)a(\alpha) + \sum_{\alpha=N+1}^{\infty} \chi(x-\alpha)a(\alpha)$$
$$+ \chi(x-N-1)A(N) - \chi(x+M)A(-M).$$

Thus, we obtain

$$||f - f_{M,N}||_1 \leqslant \sum_{\alpha=-\infty}^{-M} |a(\alpha)| + \sum_{\alpha=N+1}^{\infty} |a(\alpha)| + |A(N)| + |A(-M)|.$$

It follows

$$\lim_{M\to\infty,\ N\to\infty} ||f - f_{M,N}||_1 = 0.$$

Therefore, $f(x) \in S_1(\Psi)$.

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